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A MULTILOOP GENERALIZATION OF THE CIRCLE STABILITY CRITERION.(U)
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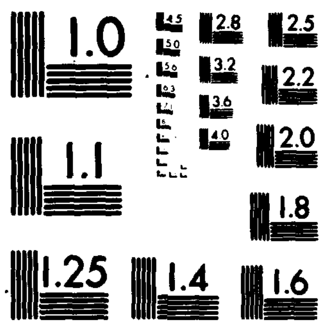

MICROCOPY RESOLUTION TEST CHART

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(6) A MULTILOOP GENERALIZATION OF THE CIRCLE STABILITY CRITERION

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Abstract

A frequency-domain stability criterion is presented, generalizing the well-known circle stability criterion to multiloop feedback systems having bounded nonlinearity, parameter variations, and/or frequency-dependent ignorance of component dynamics. Unlike previous generalizations, the theory is not restricted to weakly-coupled, diagonally dominant or nearly normal systems. Potential applications include the analysis of feedback system integrity and multiloop feedback system stability margins.

1. Introduction

A key step in the synthesis of robustly stable feedback systems is the characterization of a set of feedback laws that are stabilizing for every element of the set of possible plant dynamics. This type of information is precisely what is provided for single-loop feedback systems by such input-output stability criteria as the Nyquist, Popov, and circle theorems. Indeed, the practical merit of classical feedback design procedures involving Nyquist loci, Bode plots, and Nichols charts is in a large measure directly attributable to the fact that these design procedures provide the designer with an easily interpretable charcterization of such sets of robustly stable feedback laws. Available multivariable input-output stability criteria such as Rosenbrock's multivariable Nyquist theorem [1] and Zames' conic-relation and positivity stability theorems [2] have led to useful characterizations of sets of robustly stable feedback laws for only a limited class of problems, viz. interconnections of dissipative systems [3], weakly coupled interconnections of systems [4] (including so-called "diagonally dominant" systems [5] - [7]) and "nearly normal" systems [8] (which can be viewed as vector-space isomorphisms of weakly coupled systems). It has been argued convincingly by Rosenbrock and Cook [9] that an especially useful feedback design tool would be a more general multiloop frequency-domain stability criterion that includes diagonal dominance and normality results as special cases.

The main result of the present paper is a stability result that may serve this purpose: Theorem 1 is a multiloop generalization of the circle stability criterion which does not require diagonal-dominance, weak-coupling, normality, or near normality. The result allows the frequency-domain testing of the stability of multiloop feedback systems with time-varying nonlinearities, unknown-but-bounded parameter variations, and even singular perturbations.

The following notation is used: A^T and x^T denote respectively the transpose of the matrix A and the vector x; A^* and x^* denote the complex conjugate of the matrix A^T and the vector x^T respectively; the determinant of a matrix A is denoted $\det(A)$; the Euclidian norm of a vector x is $\|x\| \triangleq \sqrt{x^* x}$; R_+ denotes nonnegative real numbers; the functional norm $\|x\|_\tau$ and inner product $<x_1, x_2>_\tau$ are defined for functions $x: R_+ \to R^n$ as

$$\|x\|_\tau \triangleq \sqrt{<x, x>_\tau}$$

where for any x_1 and x_2

$$<x_1, x_2>_\tau \triangleq \int_0^\tau x_1^T(t)\, x_2(t)\, dt.$$

The space $L_{2e}(R_+, R^n)$ is defined as

$$L_{2e}(R_+, R^n) \triangleq \left\{ x : R_+ \to R^n \,\middle|\, \|x\|_\tau < \infty \text{ for each } \tau \in R_+ \right\}.$$

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Given any matrix A, the square-roots of the eigenvalues of $A^* A$ are called the singular values of A [10, pp. 5-11]. For any non-zero Matrix A, we use the notation $\sigma_{max}(A)$ to denote the largest singular value of A and $\sigma_{min}(A)$ to denote the smallest singular value of A; singular values are always non-negative real numbers since $A^* A$ is always positive semidefinite. For hermitian A (i.e., $A = A^*$), the notation $\lambda_{max}(A)$ and $\lambda_{min}(A)$ to denote respectively the greatest and least eigenvalues of A — hermitian matrices have only real eigenvalues, so ordering of eigenvalues is always possible via the usual ordering of real numbers.

An operator is a mapping of functions into functions; for example, a dynamical system mapping inputs $x \in L_{2e}(R_+, R^n)$ into outputs $y \in L_{2e}(R_+, R^m)$ defines an operator. All operators considered in this paper are implicitly assumed to be mappings of $L_{2e}(R_+, R^{n_1})$ into $L_{2e}(R_+, R^{n_2})$ for some positive integers n_1 and n_2, an operator $\underset{\sim}{H}$ is said to be nonanticipative if

$$(\underset{\sim}{H} x_1)(t_o) = (\underset{\sim}{H} x_2)(t_o)$$

for any t_o and any pair of functions x_1 and x_2 having the property that for all $t < t_o$,

$$x_1(t) = x_2(t);$$

i.e., a non anticipative operator $\underset{\sim}{H}$ is one having the property that its instantaneous output $\underset{\sim}{H} x$ at any time t_o is independent of the values assumed by the input $x(t)$ at future times $t > t_o$. We say that an operator $\underset{\sim}{H}$ mapping signals $x \in L_{2e}(R_+, R^{n}_1)$ into signals $\underset{\sim}{H} x \in L_{2e}(R_+, R^m)$ is L_{2e} - stable[1] if there exists a constant $k < \infty$ such that for all $x \in L_{2e}(R_+, R^n)$ and $\tau \in R_+$

$$\| \underset{\sim}{H} x \|_\tau \le k \| x \|_\tau .$$

II. Problem Formulation

Our results concern the input-output stability of systems consisting of a dynamical LTI negative-feedback interconnection of m memoryless time-varying nonlinear components and n dynamical LTI components. The system's equations thus take the following form (see Fig. 1):

Memoryless nonlinear components

$$\left.\begin{aligned} y_1(t) &= h_1(x_1(t), t) \\ &\vdots \\ y_m(t) &= h_m(x_m(t), t) \end{aligned}\right\} \quad (2.1)$$

Dynamical LTI components

$$\left.\begin{aligned} Y_{m+1}(s) &= H_{m+1}(s) X_{m+n}(s) \\ &\vdots \\ Y_{m+n}(s) &= H_{m+n}(s) X_{m+n}(s) \end{aligned}\right\} \quad (2.2)$$

Dynamical LTI interconnection

$$\underline{X}(s) = -\underline{G}(s)\left(\underline{Y}(s) + \underline{V}(s)\right) + \underline{U}(s) \quad (2.3)$$

where

$$\underline{Y}(s) \triangleq \begin{bmatrix} Y_1(s) \\ \cdot \\ \cdot \\ Y_{n+m}(s) \end{bmatrix}; \quad \underline{X}(s) \triangleq \begin{bmatrix} X_1(s) \\ \cdot \\ \cdot \\ X_{n+m}(s) \end{bmatrix}; \quad \underline{U}(s) \triangleq \begin{bmatrix} U_1(s) \\ \\ \\ U_{n+m}(s) \end{bmatrix};$$

$$\underline{V}(s) \triangleq \begin{bmatrix} V_1(s) \\ \cdot \\ \cdot \\ V_{n+m}(s) \end{bmatrix}; \quad \underline{G}(s) \triangleq \begin{bmatrix} G_{1,1}(s) \cdots G_{1,(n+m)}(s) \\ \vdots \\ G_{(n+m),1}(s) \cdots G_{(n+m),(n+m)}(s) \end{bmatrix} \quad (2.4)$$

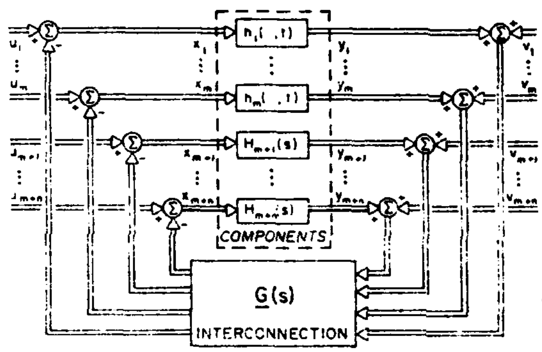

Fig. 1 The System

The indogenous variables $y_i(t)$ and $x_i(t)$ are the system 'outputs' and the exogenous variables $u_i(t)$ and $v_i(t)$ are the system 'inputs'. Each of the 'components' $(h_1(\cdot), \ldots, h_m, H_{m+1}, \ldots, H_{m+n})$ may itself be a MIMO system in general, though our results seem to be most easily used and interpreted when the components are SISO.

Our stability results do not require that we have available an exact mathematical description of the components. For each of the nonlinear nondynamical elements, we assume only that matrices C_i, R_i, and S_i can be found such that $R_i^T R_i$ and $S_i^T S_i$ are positive definite and such that

$$\left.\begin{aligned} &\| S_i\left(h_i(x_i(t), t) - C_i x_i(t)\right) \|^2 \le \| R_i x_i(t) \|^2 - \epsilon \| x_i(t) \|^2 \\ &\quad \text{for some } \epsilon > 0, \text{ all } x_i(t), \text{ and all } t \\ &(i = 1, \ldots, m). \end{aligned}\right\} \quad (2.5)$$

For each of the n dynamical LTI components $H_i(s)$ $(i = m+1, \ldots, m-n)$ we assume only

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1 A form of global input-output stability, our definition of L_{2e} - stable is equivalent to 'L^2 stable' [11] and to 'finite gain stable' (with respect to the L_2 norm) [12] when $\underset{\sim}{H}$ is a nonanticipative operator.

that $H_i(s)$ has a proper rational transfer function matrix and that proper rational transfer function matrices $C_i(s)$, $R_i(s)$, and $S_i(s)$ can be found such that $R_i^T(-j\omega)\,R_i(j\omega)$ and $S_i^T(j\omega)\,S_i(j\omega)$ are positive definite and have no poles on the $s = j\omega$ axis and such that $H_i(s) - C_i(s)$ has no poles in $\{ s \mid Re(s) \geq 0 \}$ and

$$\left.\begin{aligned} &\| S_i(j\omega)\Big(H_i(j\omega)\,X_i(j\omega) - C_i(j\omega)\,X_i(j\omega)\Big)\|^2 \\ &\leq \| R_i(j\omega)\,X_i(j\omega)\|^2 - \epsilon \|X_i(j\omega)\|^2 \\ &\text{for some } \epsilon > 0,\ \text{all } X_i(j\omega) \text{ and all } \omega \end{aligned}\right\} \quad (2.6)$$

$$(i = m+1, \ldots, m+n).$$

For notational convenience, we define the following block-diagonal matrices

$$C(s) \triangleq \mathrm{diag}\Big(C_1, \ldots, C_m, C_{m+1}(s), \ldots, C_{m+n}(s)\Big) \quad (2.7)$$

$$R(s) \triangleq \mathrm{diag}\Big(R_1, \ldots, R_m, R_{m+1}(s), \ldots, R_{m+n}(s)\Big) \quad (2.8)$$

$$S(s) \triangleq \mathrm{diag}\Big(S_1, \ldots, S_m, S_{m+1}(s), \ldots, S_{m+n}(s)\Big) \quad (2.9)$$

<u>Comments:</u>

The conditions (2.5) - (2.6) can be interpreted as saying that we are assuming knowledge about each of the components is limited to an approximate LTI model (viz. C_i) and bounds (determined by (R_i, S_i)) on the coarseness of the approximation.

For the case of SISO components, the conditions (2.5) and (2.6) can be replaced respectively by the simpler conditions

$$\left.\begin{aligned} &\frac{|h_i(x_i(t), t) - c_i x_i(t)|^2}{|x_i(t)|^2} \leq r_i^2 - \epsilon \\ &\text{for some } \epsilon > 0,\ \text{all } x_i(t) \neq 0, \text{ and all } t \end{aligned}\right\} \quad (2.5')$$

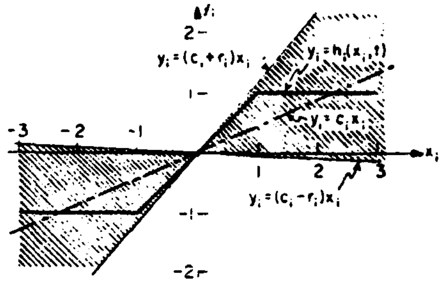
$$\left.\begin{aligned} &|H_i(j\omega) - c_i(j\omega)|^2 \leq |r_i(j\omega)|^2 - \epsilon \\ &\text{for some } \epsilon > 0 \text{ and all } \omega \end{aligned}\right\} \quad (2.6')$$

where for all $i = 1, \ldots, n+m$

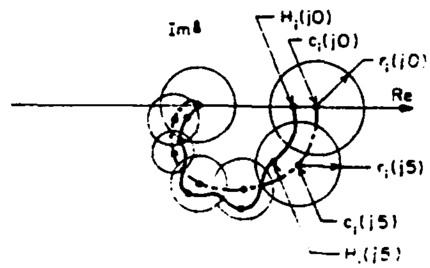
$$c_i = C_i \quad (2.10)$$

$$r_i = R_i S_i^{-1} \quad (2.11)$$

These SISO conditions are readily interpreted graphically as shown in Fig. 2.



<u>(a) Nonlinear component satisfying (2.5')</u>



<u>(b) Nyquist locus of LTI component satisfying (2.6')</u>

<u>Fig. 2 SISO Components</u>

III. <u>Main Result</u>

Our main result is now stated.

<u>Theorem I (Multiloop Circle Stability Criterion):</u>

Suppose that the system (2.1)-(2.4) is L_{2e}-stable for the case when

$$\left.\begin{aligned} h_i(x_i, t) &\equiv C_i x_i \quad (i = 1, \ldots, m) \\ H_i(s) &\equiv C_i(s) \quad (i = m+1, \ldots, m+n). \end{aligned}\right\} \quad (3.1)$$

Then, a sufficient condition for the system (2.1) - (2.4) to be L_{2e}-stable for every collection of $h_i(\cdot, t)$ $(i = 1, \ldots, m)$ and $H_i(s)$ $(i = m+1, \ldots, n)$ satisfying (2.5) and (2.6) respectively is that any one of the following conditions hold for all real ω

i) $$\lambda_{\min}\Big(\big(I + C(j\omega)G(j\omega)\big)^T S^T(-j\omega)\,S(j\omega)\big(I + C(j\omega)G(j\omega)\big) - G^T(-j\omega)R^T(-j\omega)R(j\omega)\,G(j\omega)\Big) \geq 0 \quad (3.2a)$$

ii) $$\lambda_{\min}\Big(\big(C(-j\omega) + G^{-1}(-j\omega)\big)^T S^T(-j\omega)S(j\omega)\big(C(j\omega) + G^{-1}(j\omega)\big) - R^T(-j\omega)\,R(j\omega)\Big) \geq 0 \quad (3.2b)$$

iii) $\sigma_{min}\left(S(j\omega)\left(C(j\omega) + G^{-1}(j\omega)\right)R^{-1}(j\omega)\right) \geq 1$ (3.2c)

iv) $\sigma_{max}\left(R(j\omega)\,G(j\omega)\left(I + C(j\omega)G(j\omega)\right)^{-1}S^{-1}(j\omega)\right) \leq 1$ (3.2d)

Condition (3.2a) is implied by (3.2b)-(3.2d) and, when the inverses G^{-1}, R^{-1}, S^{-1} are defined, conditions (3.2a)-(3.2d) are equivalent.

PROOF: See Appendix

IV. Discussion

There are essentially two main conditions which must be satisfied to conclude stability from Theorem 1: (i) The system must be stable when the uncertain nonlinear components $h_i(\cdot,t)$ and LTI components $H_i(s)$ are all replaced by the respective LTI approximations C_i and $C_i(s)$; and, (ii) the frequency-domain condition (3.2) must be satisfied. The former condition can be verified a variety of ways; for example, one may check that the roots of the characteristic equation

$$\det\left(I + C(s)\,G(s)\right) = 0 \qquad (4.1)$$

all have negative real parts [2]; alternatively, one may apply the multivariable Nyquist criterion, checking that the polar plot of the locus of $\det\left(C(j\omega)\,G(j\omega)\right)$ encircles the point $-1 + j0$ exactly once counterclockwise for each unstable open-loop pole of $\underline{C}(s)\ \underline{G}(s)$ (multiplicities counted) [1], [21]. The latter condition (3.2) requires that one plot the variable $\sigma_{min}(\cdot)$ or $\sigma_{max}(\cdot)$ verses ω and verify that the appropriate inequality holds for all ω.

In the special case in which there is a single scalar nonlinearity $h_1(x_1,t)$ (so that $m = 1$ and $n = 0$), both of the conditions of Theorem 1 can be verified by inspection of the polar plot of $G(j\omega)$ vs ω. Stability for the special case $h_1(x_1,t) = c_1x_1$ is assured by the Nyquist stability criterion if and only if $G(j\omega)$ encircles $-1 + j0$ once for each unstable pole of $G(s)$ as ω increases from $-\infty$ to $+\infty$. Condition (3.2) becomes

$$\left| c_1 + \frac{1}{G(j\omega)} \right| > \left| r_1 \right| \qquad (4.2)$$

where

$$r_1 = R_1 S_1^{-1}$$

$$c_1 = C_1$$

or, equivalently (assuming $c_1 > 0$),

i) if $c_1^2 - r_1^2 > 0$

$$\left| G(j\omega) + \frac{c_1}{c_1^2 - r_1^2} \right| > \left| \frac{r_1}{c_1^2 - r_1^2} \right| \qquad (4.3a)$$

ii) if $c_1^2 - r_1^2 < 0$

$$\left| G(j\omega) + \frac{c_1}{c_1^2 - r_1^2} \right| < \left| \frac{r_1}{r_1^2 - c_1^2} \right| \qquad (4.3b)$$

iii) if $c_1^2 - r_1^2 = 0$

$$\mathrm{Re}\left(G(j\omega)\right) > \frac{-1}{c_1 + |r_1|} \qquad (4.3c)$$

These conditions on $G(j\omega)$ are precisely the conditions of the well-known circle stability criterion (cf. [15]). It is this which motivates us to refer to Theorem 1 as a 'multiloop circle stability criterion' —— despite the fact that in general no circles are employed in verifying its conditions.

One can interpret the uncertainty bounds (R_i, S_i) as specifications for the gain margins and phase margins of the system (2.1)-(2.4). If $m = 0$, if $H_i(s) \equiv C_i(s)$ $(i = 1,\ldots,n)$ and if the components are SISO, then under the conditions of Theorem 1, the system will remain stable despite variations in the individual component gains of magnitudes as great as $|r_i(j\omega)| \triangleq |R_i(j\omega)/S_i(j\omega)|$, <u>even when the variations occur simultaneously in all components.</u> So, for example, the system can tolerate simultaneous gain variations or phase variations of <u>at least</u>

$$G_{M_i} \triangleq \inf_{\omega} 20 \log \left| \frac{r_i(j\omega)}{C_i(j\omega)} \right|, \text{ db} \qquad (4.4)$$

or

$$\theta_{M_i} \triangleq \inf_{\omega} \arcsin \left| \frac{r_i(j\omega)}{C_i(j\omega)} \right| \qquad (4.5)$$

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2 If $C(s)\ G(s)$ has any 'decoupling zeroes' (i.e., uncontrollable or unobservable poles), then these will not be roots of (4.1) and one must check separately that these poles have negative real parts —— cf. [14]

in each of the respective component input channel's ($i=1,\ldots,n$); i.e., the system has gain margins of at least G_{M_i} and phase margins of at least θ_{M_i} at the inputs to the respective components $C_i(s)$ ($i=1,\ldots,n$). The quantity

$$k_m \triangleq \sigma_{min}\left(S(j\omega)\left(C(j\omega)+G^{-1}(j\omega)\right)R^{-1}(j\omega)\right) \tag{4.6}$$

is the amount by which the uncertainty bounding matrices R_i can be simultaneously increased without violating the stability conditions of Theorem 1 ——the scalar k_m can be viewed as a lower bound on the amount by which the system (2.1)-(2.4) exceeds the stability margin specifications (2.5)-(2.6).

In general, the stability conditions —— and the estimate (4.6) of excess stability margin k_m —— of Theorem 1 will be conservative. The conservativeness can usually be reduced by multiplying equations (2.5)-(2.6) by appropriately chosen positive scalars $|\alpha_i|^2$ and $|\alpha_i(j\omega)|^2$ respectively before applying Theorem 1. This has the net effect of substituting 'weighted' uncertainty bounding matrices $(\alpha_i R_i, \alpha_i S_i)$ for the original matrices (R_i, S_i). Further study is required to determine the extent to which it will be practical to exploit such weighting to reduce the conservativeness of Theorem 1.

V. Conclusions

The practical importance of our multiloop circle theorem is that it provides verifiable sufficient conditions for the stability of multiloop feedback systems using only crude bounds on system parameters, component frequency responses, and nonlinearities. Potential applications include the testing of system integrity in the presence of actuator and/or sensor failures (cf. [16]) and the characterization of the stability margins (e.g., gain & phase margin) of multiloop feedback designs subject to simultaneous perturbations in gain and phase in the feedback loops. Theorem 1 also plays a key role in bounding the response of systems with uncertain dynamics [22].

Appendix

In this appendix, Theorem 1 is proved using the sector stability criterion of [17], [18], [19]. We begin by introducing some additional terminology, and a relevant special case of the sector stability criterion, viz. Theorem A1. We then establish via several lemmas that the conditions of Theorem 1 are sufficient to ensure that the conditions of Theorem A1 are satisfied.

Definition(L_{2e}-Cone$(\cdot,\cdot,\cdot)$; strictly inside, outside):

Given any three operators $\underset{\sim}{C}$, $\underset{\sim}{R}$, $\underset{\sim}{S}$, we define

$$L_{2e}-\text{Cone}(\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S}) \triangleq \left\{(\underset{\sim}{x}, \underset{\sim}{y}) \,\middle|\, \underset{\sim}{F}(x,y,\tau) \le 0 \text{ for all } \tau \in R_+\right\} \tag{A1}$$

where

$$\underset{\sim}{F}(\underset{\sim}{x},\underset{\sim}{y},\tau) \triangleq \|\underset{\sim}{S}(y-\underset{\sim}{C}x\|^2 - \|\underset{\sim}{R}x\|^2 \tag{A2}$$

and for all z, z_1, z_2

$$\|z\|_\tau \triangleq \sqrt{<z, z>_\tau} \tag{A3}$$

$$<z_1, z_2>_\tau \triangleq \int_0^\tau z_1^T(t)\, z_2(t)\, dt \tag{A4}$$

Given an operator $\underset{\sim}{H}$ mapping signals x into signals y, we say

$$\underset{\sim}{H} \text{ strictly inside } L_{2e}-\text{Cone}(\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S}) \tag{A5}$$

if there exists an $\epsilon > 0$ such that for every pair (x,y) satisfying $y = Hx$

$$\|\underset{\sim}{S}(y-\underset{\sim}{C}x)\|_\tau^2 \le \|\underset{\sim}{R}x\|_\tau^2 - \epsilon\left(\|x\|_\tau^2 + \|y\|_\tau^2\right) \tag{A6}$$

for all $\tau \in R_+$.

Given an operator $-\underset{\sim}{G}$ mapping signals y into signals x, we say

$$(-\underset{\sim}{G})^I \text{ outside } L_{2e}-\text{Cone}(\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S}) \tag{A7}$$

if for every pair (x,y) satisfying $x = -\underset{\sim}{G}y$ we have

$$\|\underset{\sim}{S}(y-\underset{\sim}{C}x)\|_\tau^2 \ge \|\underset{\sim}{R}x\|_\tau^2 \tag{A8}$$

for all $\tau \in R_+$.

Theorem A1 Let p be a positive integer; let $\underset{\sim}{H}_i$ ($i=1,\ldots,p$) be operators mapping x_i into y_i; let $\underset{\sim}{H}$ be the operator

$$\underset{\sim}{H}\underline{y} = \begin{bmatrix} \underset{\sim}{H}_1 y_1 \\ \cdot \\ \cdot \\ \underset{\sim}{H}_p y_p \end{bmatrix} \qquad (A9)$$

where

$$\underline{y} = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_p \end{bmatrix} . \qquad (A10)$$

If there exist operators $\underset{\sim}{C}_i$, $\underset{\sim}{R}_i$, $\underset{\sim}{S}_i$ $(i=1,\ldots,p)$ such that

$$\underset{\sim}{H}_i \text{ \underline{strictly inside} } L_{2e} - \text{Cone}(\underset{\sim}{C}_i, \underset{\sim}{R}_i, \underset{\sim}{S}_i)$$

$$\text{for all } i = 1,\ldots,p \qquad (A11)$$

and

$$(-\underset{\sim}{G})^I \text{ \underline{outside} } L_{2e} - \text{Cone}(\underset{\sim}{C}_i, \underset{\sim}{R}_i, \underset{\sim}{S}_i) , \qquad (A12)$$

then the system

$$\left.\begin{aligned} \underline{y} &= \underset{\sim}{H}\,\underline{x} \\ \underline{x} &= -\underset{\sim}{G}(\underline{y} + \underline{v}) + \underline{u} \end{aligned}\right\} \qquad (A13)$$

is L_{2e} stable.

Proof: The expression (A2) can be written equivalently,

$$\underset{\sim}{F}(x, y, \tau) = \langle \underset{\sim}{F}_{11} y + \underset{\sim}{F}_{12} x , \underset{\sim}{F}_{21} y + \underset{\sim}{F}_{22} x \rangle_\tau \qquad (A14)$$

where

$$\underset{\sim}{F}_{11} = \underset{\sim}{F}_{21} = \underset{\sim}{S} \qquad (A15)$$

$$\underset{\sim}{F}_{12} = -\underset{\sim}{C} + \underset{\sim}{R} \qquad (A16)$$

$$\underset{\sim}{F}_{22} = -\underset{\sim}{C} - \underset{\sim}{R} . \qquad (A17)$$

Thus

$$\text{Cone}(C_i, R_i, S_i) = \text{Sector}\begin{pmatrix} \underset{\sim}{S}_i & -\underset{\sim}{C}_i - \underset{\sim}{R}_i \\ \underset{\sim}{S}_i & -\underset{\sim}{C}_i + \underset{\sim}{R}_i \end{pmatrix}; \qquad (A18)$$

$(i = 1,\ldots, m+n)$

and

$$\text{Cone}(\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S}) = \text{Sector}\begin{pmatrix} \underset{\sim}{S} & -\underset{\sim}{C} - \underset{\sim}{R} \\ \underset{\sim}{S} & -\underset{\sim}{C} + \underset{\sim}{R} \end{pmatrix} . \qquad (A19)$$

By the composite system property of sectors (cf., Lemma 6.1 (vi) of [18]) and (A11), it follows that

$$\underset{\sim}{H} \text{ \underline{strictly inside} Sector} \begin{pmatrix} \underset{\sim}{S} & -\underset{\sim}{C} - \underset{\sim}{R} \\ \underset{\sim}{S} & -\underset{\sim}{C} + \underset{\sim}{R} \end{pmatrix} . \qquad (A20)$$

By (A12),

$$\underset{\sim}{G} \text{ \underline{outside} Sector} \begin{pmatrix} \underset{\sim}{S} & -\underset{\sim}{C} - \underset{\sim}{R} \\ \underset{\sim}{S} & -\underset{\sim}{C} + \underset{\sim}{R} \end{pmatrix} . \qquad (A21)$$

Theorem A1 follows from Theorem 6.1 of [18] (the sector stability criterion). $\square$

Theorem A1 together with the following three Lemmas, establish Theorem 1.

Lemma A2: Let $h(x(t),t)$ be any function of $x(t)$ and t and let $\underset{\sim}{H}$ be given by

$$(\underset{\sim}{H}\, x)(t) = h(x(t), t) . \qquad (A22)$$

Let C, R, and S be matrices and let $\underset{\sim}{C}$, $\underset{\sim}{R}$, $\underset{\sim}{S}$ be the operators defined by

$$(\underset{\sim}{C}x)(t) = C\, x(t) \quad \forall x \qquad (A23)$$

$$(\underset{\sim}{R}x)(t) = R\, x(t) \quad \forall x \qquad (A24)$$

$$(\underset{\sim}{S}y)(t) = S\, y(t) . \quad \forall y . \qquad (A25)$$

Suppose S^{-1} exists, then

$$\underset{\sim}{H} \text{ strictly inside } L_{2e} - \text{Cone}(\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S}) \qquad (A26)$$

if and only if

$$\left\| S\left(h(x(t), t) - C\,x(t)\right) \right\|^2 \le \left\| R\,x(t) \right\|^2 - \varepsilon \left\| x(t) \right\|^2$$

$$\forall\, x(t). \qquad (A27)$$

Proof: Let $y(t) = h(x(t),t)$. Suppose (A27) holds. Then,

$$\| y(t) \| \le \alpha \| x(t) \| \qquad (A28)$$

where

$$\alpha = \left(\frac{\sigma_{max}(R)}{\sigma_{min}(S)} + \sigma_{max}(C) \right) . \qquad (A29)$$

Thus, taking

$$\varepsilon' = \frac{\varepsilon}{1 + \alpha^2} \qquad (A30)$$

we have that when (A27) holds, then

$$\|\underset{\sim}{S}(y-\underset{\sim}{C}x)\|_T^2$$
$$= \int_0^T \| S\left(h(x(t), t)\right) - C\,x(t)\|^2\,dt$$
$$\le \int_0^T \|R\,x(t)\|^2 - \varepsilon\,\|x(t)\|^2\,dt$$
$$\le \int_0^T \|R\,x(t)\|^2 - \varepsilon'\left(\|x(t)\|^2 + \|y(t)\|^2\right)dt$$
$$= \|Rx\|_T^2 - \varepsilon'\left(\|x\|_T^2 + \|y\|_T^2\right). \tag{A31}$$

Conversely, when (A27) holds, then taking $x(t)$ to be the constant function $x(t) \equiv x_0$ we have that for some $\delta > 0$

$$\|S\left(h(x_0, t_0) - C\right)\|^2$$
$$= \frac{1}{T}\|\underset{\sim}{S}(\underset{\sim}{H}x - Cx)\|_T^2$$
$$\le \frac{1}{T}\left(\|Rx\|_T^2 - \delta\left(\|x\|_T^2 + \|y\|_T^2\right)\right)$$
$$\le \frac{1}{T}\left(\|Rx\|_T^2 - \delta\|x\|_T^2\right)$$
$$= \|Rx_0\|^2 - \delta\|x_0\|^2. \tag{A32}$$
$\square$

Lemma A3: Let $\underset{\sim}{H}, \underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S}$ be nonanticipative L_{2e}-stable linear-time-invariant operators with respective rational transfer function matrices $H(s)$, $C(s)$, $R(s)$, $S(s)$. Suppose that $R^{-1}(s)$ exists and has no poles in $Re(s) \ge 0$.

Then

$$\underset{\sim}{H} \text{ strictly inside } L_{2e}-\text{Cone}(\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S}) \tag{A33}$$

if and only if

$$\|S(j\omega)\left(H(j\omega) - C(j\omega)\right)X(j\omega)\|^2 \le \|R(j\omega)X(j\omega)\|^2 - \varepsilon\|X(j\omega)\|^2 \tag{A34}$$

for all $X(j\omega)$ all ω, and some $\varepsilon > 0$

Proof: Let $\underset{\sim}{R}^{-1}$ denote the stable nonanticipative LTI operator having transfer function matrix $R^{-1}(s)$. Suppose that (A34) holds and let

$$z_T(t) = \begin{cases} (\underset{\sim}{R}x)(t), & \text{if } t \le T \\ 0, & \text{if } t > T \end{cases} \tag{A35}$$

and let $Z_T(j\omega)$ denote the Fourier transform of $z_T(t)$. Then, for all $y = \underset{\sim}{H}x$ we have

$$\|\underset{\sim}{S}(y - \underset{\sim}{C}x)\|_T^2 = \|\underset{\sim}{S}(\underset{\sim}{H}x - \underset{\sim}{C}x)\|_T^2$$
$$= \|\underset{\sim}{S}(\underset{\sim}{H} - \underset{\sim}{C})x\|_T^2$$
(by linearity)
$$= \|\underset{\sim}{S}(\underset{\sim}{H} - \underset{\sim}{C})\underset{\sim}{R}^{-1}\underset{\sim}{R}x\|_T^2$$
(since $\underset{\sim}{R}^{-1}$ exists)
$$= \|\underset{\sim}{S}(\underset{\sim}{H} - \underset{\sim}{C})\underset{\sim}{R}^{-1}z_T\|_T^2$$
(by nonanticipativeness)
$$\le \int_0^\infty \|(\underset{\sim}{S}(\underset{\sim}{H} - \underset{\sim}{C})\underset{\sim}{R}^{-1}z_T)(t)\|^2\,dt$$
(the integral exists since $\underset{\sim}{S}, \underset{\sim}{H}, \underset{\sim}{C}$, and $\underset{\sim}{R}^{-1}$ are stable)
$$= \int_{-\infty}^{\infty} \|S(j\omega)(H(j\omega) - C(j\omega))R^{-1}(j\omega)Z_T(j\omega)\|^2\,d\omega$$
(by Parseval's Theorem)
$$\le \int_{-\infty}^{\infty} \|Z_T(j\omega)\|^2 - \varepsilon\|R^{-1}(j\omega)Z_T(j\omega)\|^2\,d\omega$$
(by (A34))
$$= \int_0^\infty \|z_T(t)\|^2 dt - \varepsilon\int_0^\infty \|(\underset{\sim}{R}^{-1}z_T)(t)\|^2\,dt$$
(by Parseval's Theorem)
$$\le \int_0^T \|z_T(t)\|^2 dt - \varepsilon\int_0^T \|(\underset{\sim}{R}^{-1}z_T)(t)\|^2\,dt$$
$$= \|z_T\|_T^2 - \varepsilon\|\underset{\sim}{R}^{-1}z_T\|_T^2$$
$$= \|\underset{\sim}{R}x\|_T^2 - \varepsilon\|\underset{\sim}{R}^{-1}\underset{\sim}{R}x\|_T^2$$
(by nonanticipativeness of $\underset{\sim}{R}^{-1}$)
$$= \|\underset{\sim}{R}x\|_T^2 - \varepsilon\|x\|_T^2$$
$$\le \|Rx\|_T^2 - \varepsilon'\left(\|x\|_T^2 + \|\underset{\sim}{H}x\|_T^2\right). \tag{A36}$$

where

$$\varepsilon' = \frac{\varepsilon}{1+\alpha^2} \tag{A37}$$

and

$$\alpha = \sup_{x,T}\left(\frac{\|\underset{\sim}{H}x\|_T}{\|x\|_T}\right) < \infty \tag{A38}$$

(since $\underset{\sim}{H}$ is stable)

Conversely suppose (A33) holds. Let X_0 and ω_0 be arbitrary. Then, letting $x(t) \to X_0 e^{j\omega_0 t}$ and $T \to \infty$, we have trivially from Parseval's Theorem that

$$\| S(j\omega_0)\,\big(H(j\omega_0) - C(j\omega_0)\big)X_0 \|$$
$$\le \|R(j\omega_0)\,X_0\|^2 - \varepsilon\,\big(\|X_0\|^2 + \|H(j\omega_0)\,X_0\|^2\big)$$
$$\le \|R(j\omega_0)\,X_0\|^2 - \varepsilon\,\|X_0\|^2 \,. \qquad \text{(A39)}$$

$\Box$

Lemma A4: Let G, C, R, S be linear-time-invariant operators with respective proper rational transfer functions $H(s)$, $C(s)$, $R(s)$, $S(s)$. Suppose that $S^{-1}(s)$ exists and has a proper rational transfer function matrix with no poles in $\mathrm{Re}(s)\ge 0$. Suppose that R, $G(I+CG)^{-1}$, and S^{-1} are L_{2e} stable and nonanticipative. Then,

$$(-G)^I \ \underline{\text{outside}}\ L_{2e} - \text{Cone}\,(C,\ R,\ S) \qquad \text{(A40)}$$

if and only if any one of the following conditions hold for all real ω

i) $$\lambda_{\min}\Big(\big(I+C(-j\omega)\,G(-j\omega)\big)^T S^T(-j\omega)\,S(j\omega)\,\big(I+C(j\omega)G(j\omega)\big)^{-1} - G^T(-j\omega)\,R^T(-j\omega)\,R(j\omega)\,G(j\omega)\Big) \ge 0 \qquad \text{(A41a)}$$

ii) $$\lambda_{\min}\Big(\big(C(-j\omega)+G^{-1}(j\omega)\big)^T S^T(-j\omega)\,S(j\omega)\big(C(j\omega)+G^{-1}(j\omega)\big) - R^T(-j\omega)\,R(j\omega)\Big) \ge 0 \qquad \text{(A41b)}$$

iii) $$\sigma_{\min}\Big(S(j\omega)\big(C(j\omega)+G^{-1}(j\omega)\big)\,R^{-1}(j\omega)\Big) \ge 1 \qquad \text{(A41c)}$$

iv) $$\sigma_{\max}\Big(R(j\omega)\,G(j\omega)\big(I+C(j\omega)\,G(j\omega)\big)^{-1} S^{-1}(j\omega)\Big) \le 1 \,. \qquad \text{(A41d)}$$

When $G^{-1}(j\omega)$ and $R^{-1}(j\omega)$ exist, conditions (A41a)-(A41d) are equivalent.

Proof:

It is trivial to show that (A41a) is always implied by (A41b) - (A41d) and that, when $G^{-1}(j\omega)$ and $R^{-1}(j\omega)$ exist, (A41a) - (A41d) are equivalent.

Suppose that (A41a) holds. Let (x, y) by any input-output pair satisfying $x = -Gy$; let

$$v = S(y - Cx) \qquad \text{(A42)}$$

and let

$$v_\tau = \begin{cases} v(t)\ , & \text{if} \quad 0 \le t \le \tau \\ 0\ , & \text{otherwise}\,. \end{cases} \qquad \text{(A43)}$$

Let $V_\tau(j\omega)$ denote the Fourier transform of v_τ. Note that from (A41a), it follows that for all $V_\tau(j\omega)$

$$\|V_\tau(j\omega)\|^2 - \|R(j\omega)\,G(j\omega)\big(I + C(j\omega)\,G(j\omega)\big)^{-1} S^{-1}(j\omega)\,V_\tau(j\omega)\|^2 \ \ge\ 0\,. \qquad \text{(A44)}$$

Now,

$$\|Ry\|_\tau^2 = \|RG(I+CG)^{-1}S^{-1}\,v\|_\tau^2$$
$$= \|RG(I+CG)^{-1}S^{-1}\,v_\tau\|_\tau^2$$

(by the nonanticipativeness of R, $G(I+CG)^{-1}$, S^{-1})

$$\le \int_0^\infty \|\big(RG(I+CG)^{-1}S^{-1}v_\tau\big)(t)\|^2\,dt$$
$$= \int_{-\infty}^{\infty} \|R(j\omega)\,G(j\omega)\big(I+C(j\omega)\,G(j\omega)\big)^{-1}S^{-1}(j\omega)\,V_\tau(j\omega)\|^2 d\omega$$

by Parseval's Theorem and the hypotheses that (R, $G(I+CG)^{-1}$, and S^{-1}) are L_{2e} stable.

$$\le \int_{-\infty}^{\infty} \|V_\tau(j\omega)\|^2\,d\omega$$
$$= \|v_\tau\|_\tau^2 = \|v\|_\tau^2$$
$$= \|S(y - Cx)\|_\tau^2 \,. \qquad \text{(A45)}$$

Conversely, suppose that (A40) holds. Let Y_0 and ω_0 be arbitrary. The letting $y \to Y_0 e^{j\omega_0 t}$ and $\tau \to \infty$ we have from (A40) and Parseval's Theorem that

$$\|R(j\omega_0)\,G(j\omega_0)\,Y_0\|^2 \le \|S(j\omega_0)\,(I + C(j\omega_0)\,G(j\omega_0))\,Y_0\|^2$$

and hence (A41a) holds. $\Box$

This completes the proof of Theorem 1.

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